MATH1001 Lab 4 (Unit 4)

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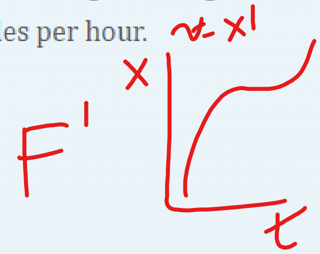
Problem 1

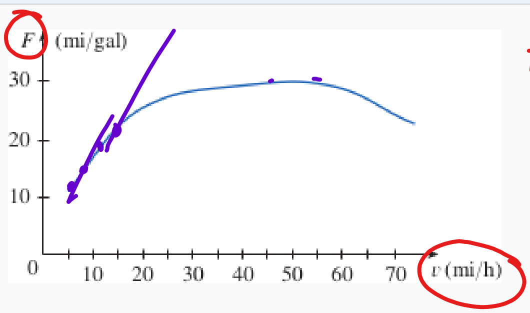
The graph (from the US Department of Energy) shows how driving speed affects gas mileage. Fuel economy F is measured in miles per gallon and speed v is measured in miles per hour.

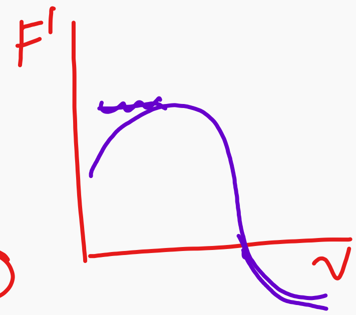
a. What is the meaning of the derivative $F'(v)$?

b. Sketch the graph of $F'(v)$.

c. At what speed should you drive if you want to save on gas?







Solution

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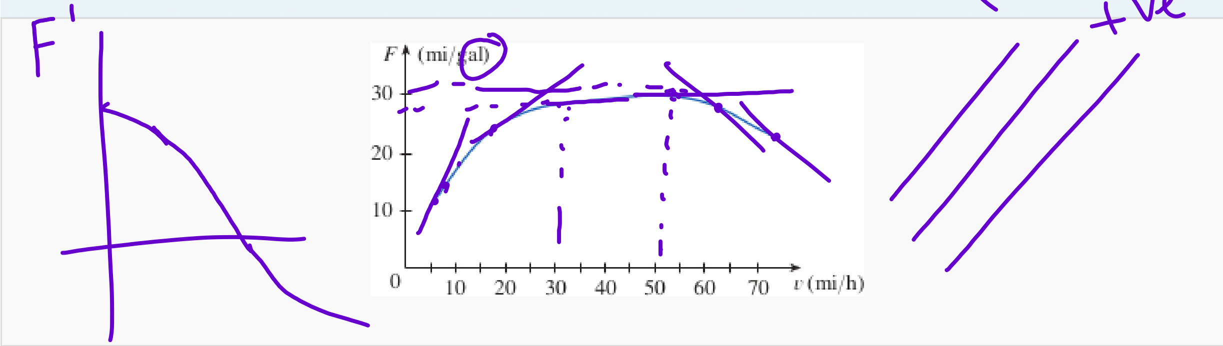
−ve

+ve

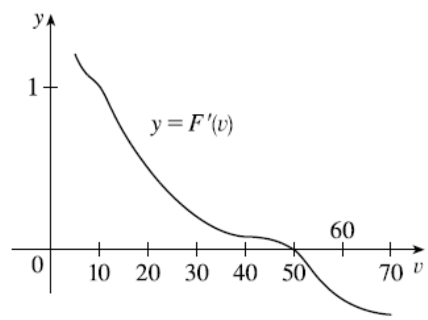


Solution

(a) $F'(v)$ is the instantaneous rate of change of fuel economy with respect to speed.

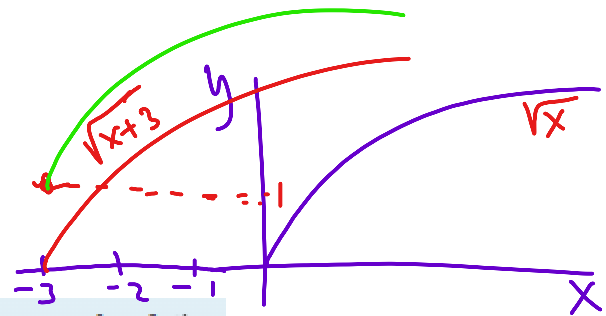
(b) Graphs will vary depending on estimates of F', but will change from positive to negative at about $v = 50$.

(c) To save on gas, drive at the speed where F is a maximum and F' is 0, which is about 50 mi/h.



Problem 2

$f(x)=\sqrt{x}$



Address the following:

$[-3,\infty) = D_f$

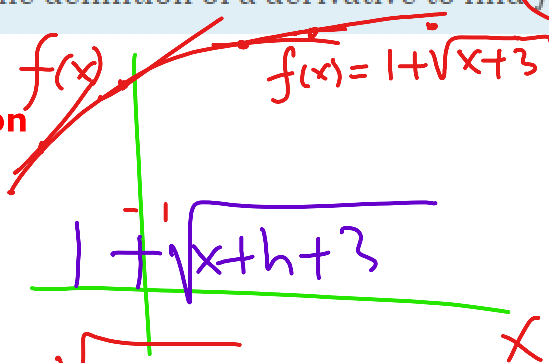
a. Sketch the graph of $f(x) = 1 + \sqrt{x+3}$

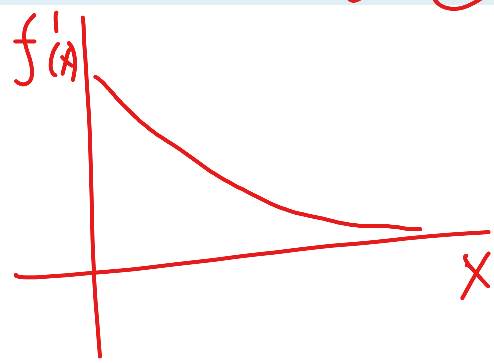
b. Use the graph from part (a) to sketch the graph of f'.

$(-3,\infty)$

c. Use the definition of a derivative to find $f'(x)$. What are the domains of f and f'?

Solution





$$f(x+h) = 1+\sqrt{x+h+3}$$

$$f(x) = 1+\sqrt{x+3}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1+\sqrt{(x+h)+3}-1-\sqrt{x+3}}{h}$$

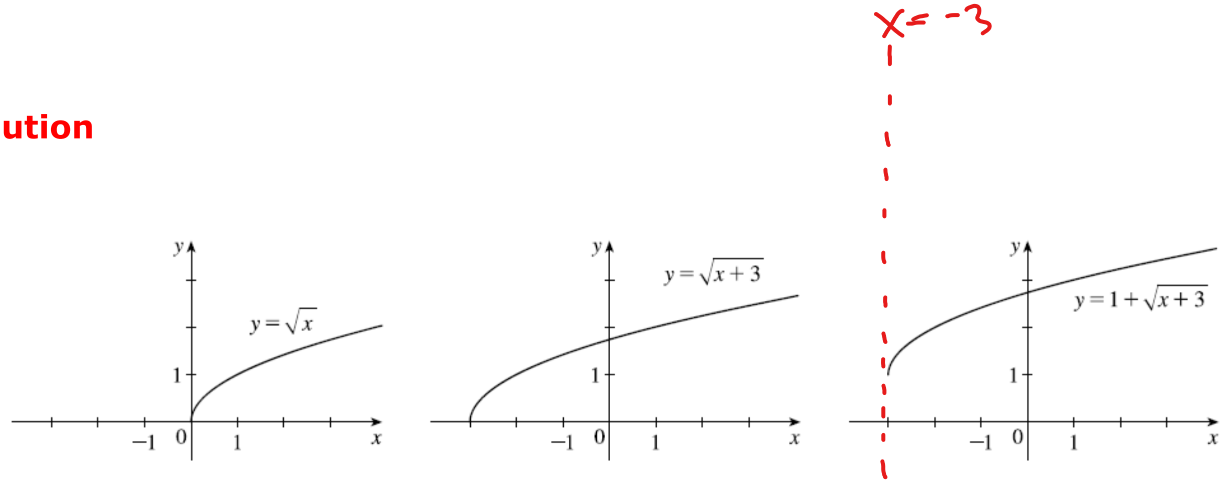
$$= \lim_{h\to 0} \frac{\sqrt{x+h+3}-\sqrt{x+3}}{h} \cdot \frac{\sqrt{x+h+3}+\sqrt{x+3}}{\sqrt{x+h+3}+\sqrt{x+3}}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+h+\cancel{3}-\cancel{x}-\cancel{3}}{h\left[\sqrt{x+h+3}+\sqrt{x+3}\right]} = \lim_{h\to 0} \frac{1}{\left(\sqrt{x+h+3}+\sqrt{x+3}\right)}$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{x+3}}}$$

Solution

(a)



(b) Note that the third graph in part (a) generally has small positive values for its slope, f'; but as $x \to -3^+$, $f' \to \infty$. See the graph in part (d).

(c) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1+\sqrt{(x+h)+3}-\left(1+\sqrt{x+3}\right)}{h}$$

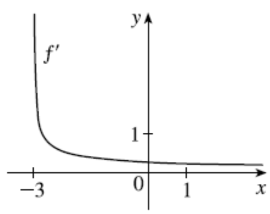
$$= \lim_{h\to 0} \frac{\sqrt{(x+h)+3}-\sqrt{x+3}}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)+3}-\sqrt{x+3}}{h}\left[\frac{\sqrt{(x+h)+3}+\sqrt{x+3}}{\sqrt{(x+h)+3}+\sqrt{x+3}}\right]$$

$$= \lim_{h\to 0} \frac{[(x+h)+3]-(x+3)}{h\sqrt{(x+h)+3}+\sqrt{x+3}} = \lim_{h\to 0} \frac{x+h+3-x-3}{h\left(\sqrt{(x+h)+3}+\sqrt{x+3}\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{(x+h)+3}+\sqrt{x+3}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{(x+h)+3}+\sqrt{x+3}} = \frac{1}{2\sqrt{x+3}}$$

Domain of $f = [-3, \infty)$, Domain of $f' = (-3, \infty)$.

(d)



Problem 3

If $f(x) = x^4 + 2x$, find $f'(x)$.

$$f(x) = x^4 + 2x$$
$$f'(x) = ?$$

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^4 + 2(x+h) - x^4 - 2x}{h}$$

$$(x+h)^2 = x^2 + h^2 + 2xh$$

$$(x+h)^4 = (x+h)^2 (x+h)^2$$

$$= (x^2 + h^2 + 2xh)(x^2 + h^2 + 2xh)$$

$$=$$

Solution

$$\text{(a)}\ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^4 + 2(x+h)] - (x^4 + 2x)}{h}$$

$$= \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 2x + 2h - x^4 - 2x}{h}$$

$$= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 2h}{h} = \lim_{h \to 0} \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3 + 2)}{h}$$

$$= \lim_{h \to 0} (4x^3 + 6x^2h + 4xh^2 + h^3 + 2) = 4x^3 + 2$$

$3^4 = 3^2 \cdot 3^2$
$= 3^3 \cdot 3$

Problem 4

$f'(x) = ?$ If $f(x) = 2x^2 - x^3$, find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$.

Solution

$$f(x) = 2x^2 - x^3$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h)^2 - (x+h)^3 - 2x^2 + x^3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x^2} + 2h^2 + 4xh - x^3 - 3h^2x - 3x^2h - h^3 - \cancel{2x^2} + x^3}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 + 4xh - 3h^2x - 3x^2h - h^3}{h} = \lim_{h \to 0} (2h + 4x - 3hx - 3x^2 - h^2)$$

$$= 4x - 3x^2$$

$$(x+h)^3 = (x+h)^2(x+h)$$
$$= (x^2 + h^2 + 2xh)(x+h)$$
$$= x^3 + h^2x + 2x^2h + x^2h + h^3 + 2xh^2$$

$$\lim_{t \to 0} \left(\frac{\sqrt{t^2+9} - 3}{t^2}\right) \cdot \left(\frac{\sqrt{t^2+9} + 3}{\sqrt{t^2+9} + 3}\right)$$

$$\lim_{t \to 0} \frac{\cancel{t^2} + \cancel{9 - 9}}{\cancel{t^2}\left(\sqrt{t^2+9} + 3\right)}$$

$$\lim_{t \to 0} \frac{1}{\sqrt{t^2+9} + 3} = \frac{1}{3+3} = \frac{1}{6}$$

$$(a+b)(a-b) = a^2 - b^2$$

$$(a-b)(a-b) = a^2 + b^2 - 2ab$$

Solution

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left[2(x+h)^2-(x+h)^3\right]-(2x^2-x^3)}{h}$$

$$= \lim_{h\to 0} \frac{h(4x+2h-3x^2-3xh-h^2)}{h} = \lim_{h\to 0}(4x+2h-3x^2-3xh-h^2) = 4x-3x^2$$

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h} = \lim_{h\to 0} \frac{\left[4(x+h)-3(x+h)^2\right]-(4x-3x^2)}{h} = \lim_{h\to 0} \frac{h(4-6x-3h)}{h}$$

$$= \lim_{h\to 0}(4-6x-3h) = 4-6x$$

$$f'''(x) = \lim_{h\to 0} \frac{f''(x+h)-f''(x)}{h} = \lim_{h\to 0} \frac{[4-6(x+h)]-(4-6x)}{h} = \lim_{h\to 0} \frac{-6h}{h} = \lim_{h\to 0}(-6) = -6$$

$$f^{(4)}(x) = \lim_{h\to 0} \frac{f'''(x+h)-f'''(x)}{h} = \lim_{h\to 0} \frac{-6-(-6)}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0}(0) = 0$$

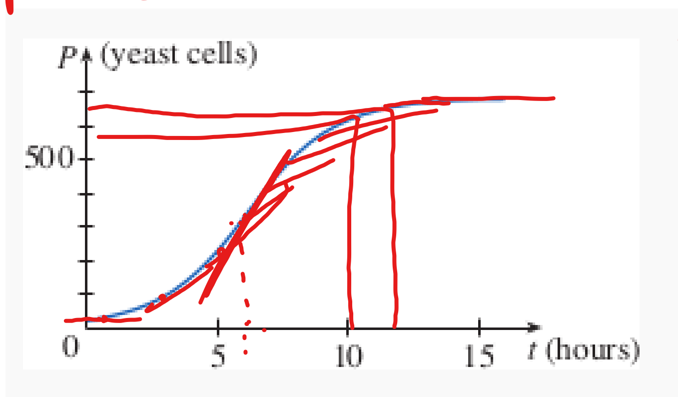
Problem 5

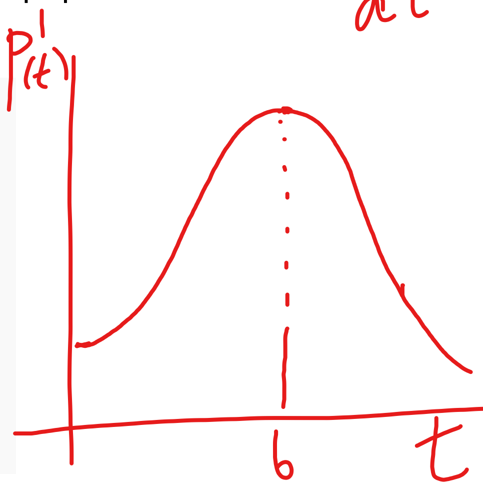
Shown is the graph of the population function *P(t)* for yeast cells in a laboratory culture.

a. Sketch the graph of the derivative *P′(t)*
b. Interpret what the graph of *P′(t)* tells about the yeast population.

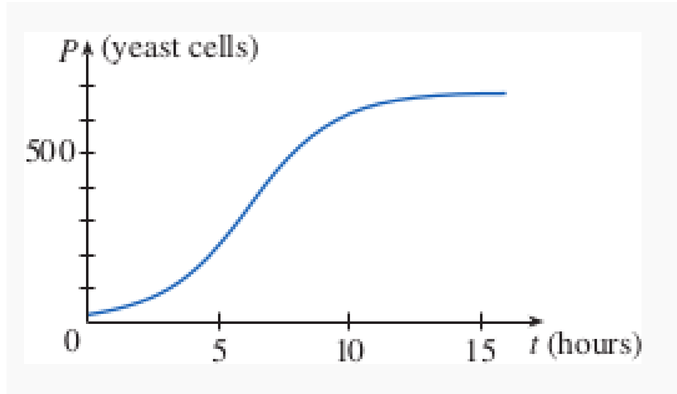
$\frac{dP}{dt}$

growth rate



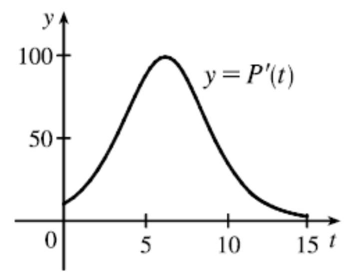


Solution



Solution

The slopes of the tangent lines on the graph of $y = P(t)$ are always positive, so the y-values of $y = P'(t)$ are always positive. These values start out relatively small and keep increasing, reaching a maximum at about $t = 6$. Then the y-values of $y = P'(t)$ decrease and get close to zero. The graph of P' tells us that the yeast culture grows most rapidly after 6 hours and then the growth rate declines.



Problem 6

Find a formula for a function that has vertical asymptotes x = 1 and x = 3, and horizontal asymptote y = 1.

Solution

$$f(x) = \frac{x^2}{(x-1)(x-3)} = \frac{x^2}{x^2-4x+3}$$

$$= \frac{1}{1-\frac{4}{x}+\frac{3}{x^2}}$$

$$(x-1)(x-3) = x^2-3x-x+3$$
$$= x^2-4x+3$$

Solution

In general if those are the only requirements, you can take a rational function,

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. Since you need 2 vertical asymptotes, you can take Q(x)=(x-1)(x-3) (in general just take a polynomial where the vertical asymptotes are the roots). Now, as for the horizontal asymptote, you can easily prove that if $deg(P) = deg(Q)$ and the leading coefficients for P and Q are p and q respectively,

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \frac{p}{q}$$

So in this example,

$$f(x) = \frac{x^2}{(x-1)(x-3)}$$

Problem 7

Prove by definition that:

$$\lim_{x\to 1}\left(\frac{2x^2-3x+1}{x-1}\right)=1$$

Solution

$$2x^2-3x+1 = 2x^2-2x-x+1$$

$$= 2x(x-1)-1(x-1)$$

$$= (x-1)(2x-1)$$

$$\lim_{x\to 1}\frac{\cancel{(x-1)}(2x-1)}{\cancel{(x-1)}} = 1$$

$\cancel{2x^2-3x+1}$

$+ \quad (-3) \quad -2, -1$

$\times \quad (2)$

Solution

$$\lim_{x \to 1}\left(\frac{2x^2 - 3x + 1}{x - 1}\right) = \lim_{x \to 1}\left(\frac{(2x-1)(x-1)}{(x-1)}\right)$$

$$= \lim_{x \to 1}(2x - 1)$$

$$= 1$$